

CONTENTS

PREFACE	ix
FOREWORD	xi
INTRODUCTION	xv

CHAPTER I

THE GAMMA FUNCTION

1.1.	Definition of the gamma function	1
1.2.	Functional equations satisfied by $\Gamma(z)$	3
1.3.	Expressions for some infinite products in terms of the gamma function	5
1.4.	Some infinite sums connected with the gamma function	7
1.5.	The beta function	9
1.5.1.	Definite integrals expressible in terms of the beta function	9
1.6.	The gamma and beta functions expressed as contour integrals.	13
1.7.	The ψ function.	15
1.7.1.	Functional equations for $\psi(z)$	16
1.7.2.	Integral representations for $\psi(z)$	16
1.7.3.	The theorem of Gauss	18
1.7.4.	Some infinite series connected with the ψ -function	19
1.8.	The function $G(z)$	20
1.9.	Expressions for the function $\log \Gamma(z)$	20
1.9.1.	Kummer's series for $\log \Gamma(z)$	23
1.10.	The generalized zeta function.	24
1.11.	The function $\Phi(z, s, \nu) = \sum_{n=0}^{\infty} (\nu+n)^{-s} z^n$	27
1.11.1.	Euler's dilogarithm	31
1.12.	The zeta function of Riemann.	32
1.13.	Bernoulli's numbers and polynomials	35
1.13.1.	The Bernoulli polynomials of higher order	39
1.14.	Euler numbers and polynomials	40

1.14.1.	The Euler polynomials of higher order	43
1.15.	Some integral formulas connected with the Bernoulli and Euler polynomials.	43
1.16.	Polygamma functions	44
1.17.	Some expansions for $\log \Gamma(1+z)$, $\psi(1+z)$, $G(1+z)$, and $\Gamma(z)$	45
1.18.	Asymptotic expansions.	47
1.19.	Mellin-Barnes integrals	49
1.20.	Power series of some trigonometric functions	50
1.21.	Some other notations and symbols	52
	References	54

CHAPTER II

THE HYPERGEOMETRIC FUNCTION

FIRST PART: THEORY

2.1.	The hypergeometric series	56
2.1.1.	The hypergeometric equation	56
2.1.2.	Elementary relations.	57
2.1.3.	The fundamental integral representations.	59
2.1.4.	Analytic continuation of the hypergeometric series.	62
2.1.5.	Quadratic and cubic transformations	64
2.1.6.	$F(a, b; c; z)$ as function of the parameters	68
2.2.	The degenerate case of the hypergeometric equation	68
2.2.1.	A particular solution.	68
2.2.2.	The full solution in the degenerate case	69
2.3.	The full solution and asymptotic expansion in the general case	74
2.3.1.	Linearly independent solutions of the hypergeometric equation in the non-degenerate case.	74
2.3.2.	Asymptotic expansions.	75
2.4.	Integrals representing or involving hypergeometric functions	78
2.5.	Miscellaneous results	81
2.5.1.	A generating function	81
2.5.2.	Products of hypergeometric series.	82
2.5.3.	Relations involving binomial coefficients and the incomplete beta function	85
2.5.4.	A continued fraction.	87
2.5.5.	Special cases of the hypergeometric function	88

2.6.	Riemann's equation.	89
2.6.1.	Reduction to the hypergeometric equation	89
2.6.2.	Quadratic and cubic transformations.	92
2.7.	Conformal representations.	93
2.7.1.	Group of the hypergeometric equation	93
2.7.2.	Schwarz's function	96
2.7.3.	Uniformization	99
2.7.4.	Zeros.	99

SECOND PART: FORMULAS

2.8.	The hypergeometric series.	101
2.9.	Kummer's series and the relations between them	105
2.10.	Analytic continuation	108
2.11.	Quadratic and higher transformations	110
2.12.	Integrals	114
	References	117

CHAPTER III

LEGENDRE FUNCTIONS

3.1.	Introduction	120
3.2.	The solutions of Legendre's differential equation	121
3.3.1.	Relations between Legendre functions.	140
3.3.2.	Some further relations with hypergeometric series	141
3.4.	Legendre functions on the cut	143
3.5.	Trigonometric expansions for $P_\nu^\mu(\cos \theta)$ and $Q_\nu^\mu(\cos \theta)$	146
3.6.1.	Special values of μ and ν	148
3.6.2.	Legendre polynomials	150
3.7.	Integral representations	155
3.8.	Relations between contiguous Legendre functions.	160
3.9.1.	Asymptotic expansions	162
3.9.2.	Behavior of the Legendre functions near the singular points.	163
3.10.	Expansions in terms of Legendre functions.	165
3.11.	The addition theorems.	168
3.12.	Integrals involving Legendre functions	169
3.13.	The ring or toroidal functions.	173
3.14.	The conical functions	174

3.15.	Gegenbauer functions	175
3.15.1.	Gegenbauer polynomials	175
3.15.2.	Gegenbauer functions	178
3.16.	Some other notations.	179
	References	180

CHAPTER IV

THE GENERALIZED HYPERGEOMETRIC SERIES

4.1.	Introduction.	182
4.2.	Differential equations	184
4.3.	Identities and recurrence relations.	185
4.4.	Generalized hypergeometric series with unit argument in the case $p = q + 1$	188
4.5.	Transformations of ${}_qF_q$ and values for arguments other than unity	190
4.6.	Integrals	192
4.7.	Various special results	192
4.8.	Basic hypergeometric series	195
	References	199

CHAPTER V

FURTHER GENERALIZATIONS OF THE HYPERGEOMETRIC
FUNCTION

5.1.	Various generalizations	202
------	-----------------------------------	-----

MACROBERT'S E -FUNCTION

5.2.	Definition of the E -function	203
5.2.1.	Recurrence relations.	205
5.2.2.	Integrals	205

MEIJER'S G -FUNCTION

5.3.	Definition of the G -function	206
5.3.1.	Simple identities	209
5.4.	Differential equations	210
5.4.1.	Asymptotic expansions.	211
5.5.	Series and integrals	213
5.5.1.	Series of G -functions	213
5.5.2.	Integrals with G -functions	214

5.6.	Particular cases of the G -function	215
------	---	-----

HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

5.7.	Hypergeometric series in two variables	222
5.7.1.	Horn's list	224
5.7.2.	Convergence of the series	227
5.8.	Integral representations	229
5.8.1.	Double integrals of Euler's type	230
5.8.2.	Single integrals of Euler's type	231
5.8.3.	Mellin-Barnes type double integrals	232
5.9.	Systems of partial differential equations	232
5.9.1.	Ince's investigation	237
5.10.	Reduction formulas	237
5.11.	Transformations.	239
5.12.	Symbolic forms and expansions	243
5.13.	Special cases.	244
5.14.	Further series	245
	References	246

CHAPTER VI

THE CONFLUENT HYPERGEOMETRIC FUNCTION

6.1.	Orientation	248
6.2.	Differential equations	249
6.3.	The general solution of the confluent equation near the origin	252
6.4.	Elementary relations for the Φ function	253
6.5.	Basic integral representations.	255
6.6.	Elementary relations for the Ψ function	257
6.7.	Fundamental systems of solutions of the confluent equation	258
6.7.1.	The logarithmic case.	260
6.8.	Further properties of the Ψ function	262
6.9.	Whittaker functions.	264
6.9.1.	Bessel functions.	265
6.9.2.	Other particular confluent hypergeometric functions	266
6.10.	Laplace transforms and confluent hypergeometric functions	269
6.11.	Integral representations	271
6.11.1.	The Φ function	271

6.11.2.	The Ψ function	273
6.11.3.	Whittaker functions	274
6.12.	Expansions in terms of Laguerre polynomials and Bessel functions	275
6.13.	Asymptotic behavior	277
6.13.1.	Behavior for large $ x $	278
6.13.2.	Large parameters	278
6.13.3.	Variable and parameters large	280
6.14.	Multiplication theorems.	282
6.15.	Series and integral formulas	283
6.15.1.	Series	283
6.15.2.	Integrals	284
6.15.3.	Products of confluent hypergeometric functions	286
6.16.	Real zeros for real a, c	288
6.17.	Descriptive properties for real a, c, x	291
	References	293
	SUBJECT INDEX	296
	INDEX OF NOTATIONS	301