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(P a constant matrix; $q(0; t) = 0$)

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(P a constant matrix; $q(0; t) = 0$)
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$$\frac{dx}{dt} = P(t)x + q(x; t)$$

($P(t)$ a variable matrix; $q(0; t) = 0$)

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