# Contents


*Preface* *page* xiii
*Acknowledgements* xvii

**Part one: Turbulence** 1

**1 Introduction** 3
1.1 Turbulence 3
1.2 Low-dimensional models 6
1.3 The contents of this book 10
1.4 Notation and mathematical jargon 13

**2 Coherent structures** 20
2.1 Introduction 20
2.2 Flows with coherent structures 26
2.3 Detection of coherent structures 39
2.4 The mixing layer 44
2.5 The turbulent boundary layer 63
2.6 A preview of things to come 82

**3 Proper orthogonal decomposition** 86
3.1 Introduction 88
3.2 On domains and averaging 91
3.3 Properties of the POD 93
3.3.1 Span of the empirical basis 94
3.3.2 Optimality 97
3.3.3 Symmetry 100
3.3.4 Attractors 103
3.4 Further results 108
3.5 Stochastic estimation 110
3.6 Coherent structures and homogeneity 113

| | | | |
|---|---|---|---|
| | 3.7 | Some applications | 117 |
| | | 3.7.1 Wall bounded flows | 117 |
| | | 3.7.2 Free shear flows | 120 |
| | | 3.7.3 Rayleigh–Bénard convection | 121 |
| | | 3.7.4 Model problems | 121 |
| | 3.8 | Appendix: some foundations | 122 |
| | | 3.8.1 Probability measures | 122 |
| | | 3.8.2 Compactness of $\mathfrak{R}$ | 124 |
| | | 3.8.3 Symmetry and invariant subspaces | 126 |
| | | 3.8.4 Spectral decay and approximate compactness | 127 |
| **4** | **Galerkin projection** | | 129 |
| | 4.1 | Introduction | 130 |
| | 4.2 | Some simple PDEs revisited | 133 |
| | 4.3 | The Navier–Stokes equations | 139 |
| | 4.4 | Towards low-dimensional models | 144 |
| | **Part two: Dynamical systems** | | 155 |
| **5** | **Qualitative theory** | | 157 |
| | 5.1 | Linearisation and invariant manifolds | 159 |
| | 5.2 | Periodic orbits and Poincaré maps | 166 |
| | 5.3 | Structural stability and genericity | 169 |
| | 5.4 | Bifurcations local and global | 174 |
| | 5.5 | Attractors simple and strange | 187 |
| **6** | **Symmetry** | | 201 |
| | 6.1 | Equivariant vector fields | 202 |
| | 6.2 | Local bifurcation with symmetry | 206 |
| | 6.3 | Global behavior with symmetry | 207 |
| | 6.4 | An $O(2)$-equivariant ODE | 215 |
| **7** | **One-dimensional "turbulence"** | | 227 |
| | 7.1 | Projection onto Fourier modes | 229 |
| | 7.2 | Local bifurcations from $u = 0$ | 231 |
| | 7.3 | The second bifurcation point | 234 |
| | 7.4 | Spatio-temporal chaos | 240 |
| **8** | **Randomly perturbed systems** | | 253 |
| | 8.1 | An Ornstein–Uhlenbeck process | 254 |
| | 8.2 | Noisy heteroclinic cycles | 258 |
| | 8.3 | Power spectra of homoclinic attractors | 267 |
| | 8.4 | Symmetry breaking | 269 |

**Part three: The boundary layer** 273

**9 Low-dimensional models** 275
9.1 Equations for coherent structures 276
9.2 The eigenfunction expansion 281
9.3 Symmetries 282
9.4 Galerkin projection 284
9.5 Geometrical structure of the model 292
9.6 Choosing subspaces and domains 294
9.7 The energy budget 299
9.7.1 The ratio $\langle u_1 u_2 \rangle / \langle u_i u_i \rangle$ 300
9.7.2 The mean velocity profile 303
9.8 Non-linear feedback 305
9.9 Interaction with unresolved modes 311

**10 Behavior of the models** 315
10.1 Backbones for the models 316
10.2 Heteroclinic cycles 321
10.3 Bursts and sweeps 326
10.4 The pressure term 328
10.5 More modes and instabilities 333
10.6 A tentative summary 338
10.7 Appendix: coefficients 344

**Part four: Other applications and related work** 347

**11 Some other fluid problems** 349
11.1 The circular jet 350
11.2 The transitional boundary layer 355
11.3 A forced transitional mixing layer 360
11.4 Flows in complex geometries 364
11.5 "Full channel" wall layer models 368
11.6 Discussion 373

**12 Review: prospects for rigor** 376
12.1 The quality of models 377
12.2 A short-time tracking estimate 381
12.3 Stability, simulations, and statistics 385
12.4 Spatial localisation 390
12.5 The utility of models 395

*Bibliography* 401
*Index* 417