Contents

I Introduction

1.1 Measurable Space; Probability Space 1
1.2 Measurable Functions; Random Variables 6
1.3 Product Space 7
1.4 Integrals 8
1.5 The Fubini–Tonelli Theorem 12
1.6 Integrals on R_1 13
1.7 Functions of Bounded Variation 16
1.8 Signed Measure; Decomposition Theorems 18
1.9 The Lebesgue Integral on R_1 20
1.10 Inequalities 24
1.11 Convex Functions 27
1.12 Analytic Functions 28
1.13 Jensen's and Carleman's Theorems 31
1.14 Analytic Continuation 32
1.15 Maximum Modulus Theorem and Theorems of Phragmén–Lindelöf . . 35
1.16 Inner Product Space 37

II Fourier Series and Fourier Transforms

2.1 The Riemann–Lebesgue Lemma 43
2.2 Fourier Series 49
2.3 The Fourier Transform of a Function in $L^1(-\infty, \infty)$ 52
2.4 Magnitude of Fourier Coefficients; the Continuity Modulus 53
2.5 More about the Magnitude of Fourier Coefficients 56
2.6 Some Elementary Lemmas 59
2.7 Continuity and Magnitude of Fourier Transforms 62
2.8 Operations on Fourier Series 64
2.9 Operations on Fourier Transforms 67
2.10 Completeness of Trigonometric Functions 69
2.11 Unicity Theorem for Fourier Transforms 73
2.12 Fourier Series and Fourier Transform of Convolutions 74
Notes 78

III Fourier–Stieltjes Coefficients, Fourier–Stieltjes Transforms and Characteristic Functions

3.1 Monotone Functions and Distribution Functions 81
3.2 Fourier–Stieltjes Series 84
3.3 Average of Fourier–Stieltjes Coefficients 86
3.4 Unicity Theorem for Fourier–Stieltjes Coefficients 88
3.5 Fourier–Stieltjes Transform and Characteristic Function 89
3.6 Periodic Characteristic Functions 93
3.7 Some Inequality Relations for Characteristic Functions 95
3.8 Average of a Characteristic Function 102
3.9 Convolution of Nondecreasing Functions 104
3.10 The Fourier–Stieltjes Transform of a Convolution and the Bernoulli Convolution 108
Notes 111

IV Convergence and Summability Theorems

4.1 Convergence of Fourier Series 113
4.2 Convergence of Fourier–Stieltjes Series 121
4.3 Fourier's Integral Theorems; Inversion Formulas for Fourier Transforms 123
4.4 Inversion Formula for Fourier–Stieltjes Transforms 128
4.5 Summability 131
4.6 (C,1)-Summability for Fourier Series 135
4.7 Abel-Summability for Fourier Series 140

4.8 Summability Theorems for Fourier Transforms 145
4.9 Determination of the Absolutely Continuous Component of a Nondecreasing Function 151
4.10 Fourier Series and Approximate Fourier Series of a Fourier–Stieltjes Transform 154
4.11 Some Examples, Using Fourier Transforms 158
Notes 164

V **General Convergence Theorems**

5.1 Nature of the Problems 166
5.2 Some General Convergence Theorems I 167
5.3 Some General Convergence Theorems II 174
5.4 General Convergence Theorems for the Stieltjes Integral 178
5.5 Wiener's Formula 181
5.6 Applications of General Convergence Theorems to the Estimates of a Distribution Function 186
Notes 192

VI L^2**-Theory of Fourier Series and Fourier Transforms**

6.1 Fourier Series in an Inner Product Space 194
6.2 Fourier Transform of a Function in $L^2(-\infty, \infty)$ 201
6.3 The Class H^2 of Analytic Functions 210
6.4 A Theorem of Szegö and Smirnov 214
6.5 The Class $\mathfrak{H}^2$ of Analytic Functions 218
6.6 A Theorem of Paley and Wiener 228
Notes 231

VII **Laplace and Mellin Transforms**

7.1 The Laplace Transform 232
7.2 The Convergence Abscissa 238
7.3 Analyticity of a Laplace–Stieltjes Transform 242
7.4 Inversion Formulas for Laplace Transforms 246
7.5 The Laplace Transform of a Convolution 252
7.6 Operations of Laplace Transforms and Some Examples 259
7.7 The Bilateral Laplace–Stieltjes Transform 265
7.8 Mellin–Stieltjes Transforms 269
7.9 The Mellin Transform 273
Notes 276

VIII More Theorems on Fourier and Laplace Transforms

8.1 A Theorem of Hardy 278
8.2 A Theorem of Paley and Wiener on Exponential Entire Functions 284
8.3 Theorems of Ingham and Levinson 288
8.4 Singularities of Laplace Transforms 298
8.5 Abelian Theorems for Laplace Transforms 301
8.6 Tauberian Theorems 304
8.7 Multiple Fourier Series and Transforms 314
8.8 Nondecreasing Functions and Distribution Functions in R_m 323
8.9 The Multiple Fourier–Stieltjes Transform 326
Notes 328

IX Convergence of Distribution Functions and Characteristic Functions

9.1 Helly Theorems and Convergence of Nondecreasing Functions 330
9.2 Convergence of Distribution Functions with Bounded Spectra 339
9.3 Convergence of Distribution Functions 341
9.4 Continuous Distribution Functions: A General Integral Transform of a Characteristic Function 353
9.5 A Basic Theorem on Analytic Characteristic Functions 356
9.6 Continuity Theorems on Intervals and Uniqueness Theorems 357
9.7 The Compact Set of Characteristic Functions 362
Notes 365

X Some Properties of Characteristic Functions

10.1 Characteristic Properties of Fourier Coefficients 366
10.2 Basic Theorems on Characterization of a Characteristic Function 372
103. Characteristic Properties of Characteristic Functions 376
10.4 Functions of the Wiener Class 383
10.5 Some Sufficient Criteria for Characteristic Functions 385
10.6 More Criteria for Characteristic Functions 392
Notes 397

XI Distribution Functions and Their Characteristic Functions

11.1 Moments, Basic Properties 400
11.2 Smoothness of a Characteristic Function and the Existence of Moments 408
11.3 More about Smoothness of Characteristic Functions and Existence of Moments 419

11.4 Absolute Moments 428
11.5 Boundedness of the Spectra of Distribution Functions 431
11.6 Integrable Characteristic Functions 437
11.7 Analyticity of Distribution Functions 439
11.8 Mean Concentration Function of a Distribution Function 445
11.9 Some Properties of Analytic Characteristic Functions 452
11.10 Characteristic Functions Analytic in the Half-Plane 456
11.11 Entire Characteristic Functions I 460
11.12 Entire Characteristic Functions II 466
Notes 473

XII **Convergence of Series of Independent Random Variables**

12.1 Convergence of a Sequence of Random Variables 475
12.2 The Borel Theorem 486
12.3 The Zero–One Law 488
12.4 The Equivalence Theorem 492
12.5 The Three Series Theorem 496
12.6 Sufficient Conditions for the Convergence of a Series 501
12.7 Convergence Criteria and the Typical Function 504
12.8 Rademacher and Steinhaus Functions 507
12.9 Convergence of Products of Characteristic Functions 513
12.10 Unconditional Convergence 519
12.11 Absolute Convergence 526
12.12 Essential Convergence 530
Notes 534

XIII **Properties of Sums of Independent Random Variables; Convergence of Series in the Mean**

13.1 Continuity and Discontinuity Properties of the Sum of a Series . . . 536
13.2 Integrability of the Sum of a Series 543
13.3 Magnitude of the Characteristic Functions of the Sums of Series . . . 546
13.4 Distribution Functions of the Sums of Rademacher Series; Characteristic Functions of Singular Distributions 556
13.5 Further Theorems on Rademacher Series 565
13.6 Sums of Independent Random Variables 573
13.7 Convergent Systems 583
13.8 Integrability of Sums of Series; Strong and Weak Convergences of Series 586
13.9 Vanishing of the Sum of a Series 590
13.10 Summability of Series 596
Notes 602

XIV Some Special Series of Random Variables

14.1 Fourier Series with Rademacher Coefficients 605
14.2 Random Fourier Series 612
14.3 Random Power Series, Convergence 619
14.4 Convergence of Random Power Series with Identically and Independently Distributed Random Coefficients 625
14.5 Analytic Continuation of Random Power Series 628
14.6 Fourier Series with Orthogonal Random Coefficients 634
Notes 640

References 642

INDEX 661