

Contents

Preface	v
Authors' Affiliations	xv
1. Introduction	1
1.1. Historical Background	1
1.2. Some Examples of Spectral Methods	3
1.2.1. A Fourier Galerkin Method for the Wave Equation. . .	3
1.2.2. A Chebyshev Collocation Method for the Heat Equation	7
1.2.3. A Legendre Tau Method for the Poisson Equation. . .	10
1.2.4. Basic Aspects of Galerkin, Tau and Collocation Methods	12
1.3. The Equations of Fluid Dynamics	13
1.3.1. Compressible Navier-Stokes	13
1.3.2. Compressible Euler	15
1.3.3. Compressible Potential	16
1.3.4. Incompressible Flow	17
1.3.5. Boundary Layer	18
1.4. Spectral Accuracy for a Two-Dimensional Fluid Calculation	19
1.5. Three-Dimensional Applications in Fluids	25
2. Spectral Approximation	31
2.1. The Fourier System	32
2.1.1. The Continuous Fourier Expansion	32
2.1.2. The Discrete Fourier Expansion	38
2.1.3. Differentiation	42
2.1.4. The Gibbs Phenomenon	45
2.2. Orthogonal Polynomials in $(-1, 1)$	53
2.2.1. Sturm–Liouville Problems	53
2.2.2. Orthogonal Systems of Polynomials	54
2.2.3. Gauss-Type Quadratures and Discrete Polynomial Transforms	55
2.3. Legendre Polynomials	60
2.3.1. Basic Formulas	60
2.3.2. Differentiation	62

2.4. Chebyshev Polynomials	65
2.4.1. Basic Formulas.....	65
2.4.2. Differentiation	68
2.5. Generalizations	70
2.5.1. Jacobi Polynomials	70
2.5.2. Mapping	71
2.5.3. Semi-Infinite Intervals.....	72
2.5.4. Infinite Intervals.....	74
3. Fundamentals of Spectral Methods for PDEs	76
3.1. Spectral Projection of the Burgers Equation	76
3.1.1. Fourier Galerkin	77
3.1.2. Fourier Collocation.....	78
3.1.3. Chebyshev Tau.....	79
3.1.4. Chebyshev Collocation	81
3.2. Convolution Sums.....	82
3.2.1. Pseudospectral Transform Methods	83
3.2.2. Aliasing Removal by Padding or Truncation.....	84
3.2.3. Aliasing Removal by Phase Shifts	85
3.2.4. Convolution Sums in Chebyshev Methods.....	86
3.2.5. Relation Between Collocation and Pseudospectral Methods	86
3.3. Boundary Conditions	87
3.4. Coordinate Singularities.....	90
3.4.1. Polar Coordinates	90
3.4.2. Spherical Polar Coordinates.....	91
3.5. Two-Dimensional Mapping.....	92
4. Temporal Discretization	94
4.1. Introduction.....	94
4.2. The Eigenvalues of Basic Spectral Operators.....	96
4.2.1. The First-Derivative Operator	96
4.2.2. The Second-Derivative Operator	98
4.3. Some Standard Schemes.....	101
4.3.1. Multistep Schemes.....	101
4.3.2. Runge–Kutta Methods.....	107
4.4. Special Purpose Schemes	110
4.4.1. High Resolution Temporal Schemes	110
4.4.2. Special Integration Techniques.....	112
4.4.3. Lerat Schemes	113
4.5. Conservation Forms	114
4.6. Aliasing.....	118
5. Solution Techniques for Implicit Spectral Equations	124
5.1. Direct Methods	125

5.1.1. Fourier Approximations	125
5.1.2. Chebyshev Tau Approximations	129
5.1.3. Schur-Decomposition and Matrix-Diagonalization.	133
5.2. Fundamentals of Iterative Methods	137
5.2.1. Richardson Iteration	137
5.2.2. Preconditioning	139
5.2.3. Non-Periodic Problems	144
5.2.4. Finite-Element Preconditioning	148
5.3. Conventional Iterative Methods	149
5.3.1. Descent Methods for Symmetric, Positive-Definite Systems	149
5.3.2. Descent Methods for Non-Symmetric Problems	155
5.3.3. Chebyshev Acceleration	157
5.4. Multidimensional Preconditioning	159
5.4.1. Finite-Difference Solvers.	159
5.4.2. Modified Finite-Difference Preconditioners	160
5.5. Spectral Multigrid Methods.	166
5.5.1. Model Problem Discussion.	166
5.5.2. Two-Dimensional Problems	168
5.5.3. Interpolation Operators	170
5.5.4. Coarse-Grid Operators	172
5.5.5. Relaxation Schemes.	172
5.6. A Semi-Implicit Method for the Navier–Stokes Equations	174
6. Simple Incompressible Flows.	183
6.1. Burgers Equation	183
6.2. Shear Flow Past a Circle	186
6.3. Boundary-Layer Flows.	188
6.4. Linear Stability	193
7. Some Algorithms for Unsteady Navier–Stokes Equations	201
7.1. Introduction.	201
7.2. Homogeneous Flows.	203
7.2.1. A Spectral Galerkin Solution Technique	203
7.2.2. Treatment of the Nonlinear Terms	204
7.2.3. Refinements	207
7.2.4. Pseudospectral and Collocation Methods.	208
7.3. Inhomogeneous Flows	212
7.3.1. Coupled Methods	213
7.3.2. Splitting Methods	222
7.3.3. Galerkin Methods	226
7.3.4. Other Confined Flows.	228
7.3.5. Unbounded Flows	230
7.3.6. Aliasing in Transition Calculations	231

7.4. Flows with Multiple Inhomogeneous Directions.	233
7.4.1. Choice of Mesh	234
7.4.2. Coupled Methods	236
7.4.3. Splitting Methods	237
7.4.4. Other Methods	238
7.5. Mixed Spectral/Finite-Difference Methods	238
8. Compressible Flow	240
8.1. Introduction	240
8.2. Boundary Conditions for Hyperbolic Problems	242
8.3. Basic Results for Scalar Nonsmooth Problems	246
8.4. Homogeneous Turbulence	252
8.5. Shock-Capturing	255
8.5.1. Potential Flow	255
8.5.2. Ringleb Flow	259
8.5.3. Astrophysical Nozzle	264
8.6. Shock-Fitting	266
8.7. Reacting Flows	273
9. Global Approximation Results	275
9.1. Fourier Approximation	275
9.1.1. Inverse Inequalities for Trigonometric Polynomials ...	275
9.1.2. Estimates for the Truncation and Best Approximation Errors	276
9.1.3. Estimates for the Interpolation Error	279
9.2. Sturm–Liouville Expansions	281
9.2.1. Regular Sturm–Liouville Problems	282
9.2.2. Singular Sturm–Liouville Problems	284
9.3. Discrete Norms	286
9.4. Legendre Approximations	287
9.4.1. Inverse Inequalities for Algebraic Polynomials	288
9.4.2. Estimates for the Truncation and Best Approximation Errors	288
9.4.3. Estimates for the Interpolation Error	293
9.5. Chebyshev Approximations	294
9.5.1. Inverse Inequalities for Polynomials	295
9.5.2. Estimates for the Truncation and Best Approximation Errors	295
9.5.3. Estimates for the Interpolation Error	298
9.5.4. Proofs of Some Approximation Results	299
9.6. Other Polynomial Approximations	305
9.6.1. Jacobi Polynomials	306
9.6.2. Laguerre and Hermite Polynomials	306
9.7. Approximation Results in Several Dimensions	307

9.7.1. Fourier Approximations	307
9.7.2. Legendre Approximations	308
9.7.3. Chebyshev Approximations	310
9.7.4. Blended Fourier and Chebyshev Approximations	311
10. Theory of Stability and Convergence for Spectral Methods	315
10.1. The Three Examples Revisited	315
10.1.1. A Fourier Galerkin Method for the Wave Equation	316
10.1.2. A Chebyshev Collocation Method for the Heat Equation	317
10.1.3. A Legendre Tau Method for the Poisson Equation	321
10.2. Towards a General Theory	323
10.3. General Formulation of Spectral Approximations to Linear Steady Problems	325
10.4. Galerkin, Collocation and Tau Methods	329
10.4.1. Galerkin Methods	330
10.4.2. Tau Methods	335
10.4.3. Collocation Methods	344
10.5. General Formulation of Spectral Approximations to Linear Evolution Equations	353
10.5.1. Conditions for Stability and Convergence: The Parabolic Case	355
10.5.2. Conditions for Stability and Convergence: The Hyperbolic Case	362
10.6. The Error Equation	371
11. Steady, Smooth Problems	375
11.1. The Poisson Equation	375
11.1.1. Legendre Methods	376
11.1.2. Chebyshev Methods	377
11.1.3. Other Boundary Value Problems	382
11.2. Advection-Diffusion Equation	383
11.2.1. Linear Advection-Diffusion Equation	383
11.2.2. Steady Burgers Equation	386
11.3. Navier–Stokes Equations	392
11.3.1. Compatibility Conditions Between Velocity and Pressure	394
11.3.2. Direct Discretization of the Continuity Equation: The “inf-sup” Condition	397
11.3.3. Discretizations of the Continuity Equation by an Influence-Matrix Technique: The Kleiser– Schumann Method	404
11.3.4. Navier–Stokes Equations in Streamfunction Formulation	406

11.4. The Eigenvalues of Some Spectral Operators	407
11.4.1. The Discrete Eigenvalues for $Lu = -u_{xx}$	407
11.4.2. The Discrete Eigenvalues for $Lu = -\nu u_{xx} + bu_x$	409
11.4.3. The Discrete Eigenvalues for $Lu = u_x$	412
12. Transient, Smooth Problems	415
12.1. Linear Hyperbolic Equations	415
12.1.1. Periodic Boundary Conditions	415
12.1.2. Non-Periodic Boundary Conditions	421
12.1.3. Hyperbolic Systems	427
12.1.4. Spectral Accuracy for Non-Smooth Solutions	430
12.2. Heat Equation	435
12.2.1. Semi-Discrete Approximation	435
12.2.2. Fully Discrete Approximation	437
12.3. Advection-Diffusion Equation	440
12.3.1. Semi-Discrete Approximation	440
12.3.2. Fully Discrete Approximation	441
13. Domain Decomposition Methods	444
13.1. Introduction	444
13.2. Patching Methods	447
13.2.1. Notation	447
13.2.2. Discretization	448
13.2.3. Solution Techniques	454
13.2.4. Examples	456
13.3. Variational Methods	459
13.3.1. Formulation	459
13.3.2. The Spectral-Element Method	461
13.4. The Alternating Schwarz Method	466
13.5. Mathematical Aspects of Domain Decomposition Methods	470
13.5.1. Patching Methods	470
13.5.2. Equivalence Between Patching and Variational Methods	471
13.6. Some Stability and Convergence Results	473
13.6.1. Patching Methods	473
13.6.2. Variational Methods	475
Appendices	477
A. Basic Mathematical Concepts	477
B. Fast Fourier Transforms	499
C. Jacobi–Gauss–Lobatto Roots	525
References	529
Index	551