



# Contents

|   |           |
|---|-----------|
| <b>1. Introduction .....</b>  | <b>1</b>  |
| 1.1 Stability Bifurcations .....  | 3         |
| 1.2 Center Manifold Theorem.....  | 8         |
| 1.3 Limit Cycles and Degenerate Hopf Bifurcations .....                               | 11        |
| <br>  |           |
| <b>2. The Hopf Bifurcation Theorem .....</b>  | <b>13</b> |
| 2.1 Introduction .....  | 14        |
| 2.2 The Hopf Bifurcation Theorem in the Time Domain .....                             | 15        |
| 2.2.1 Preliminaries .....   | 16        |
| 2.2.2 The Hopf bifurcation theorem .....  | 20        |
| 2.3 The Hopf Theorem in the Frequency Domain .....                                    | 21        |
| 2.4 Equivalence of the Two Hopf Theorems .....  | 26        |
| 2.5 Advantages of the Frequency Domain Approach .....                                 | 31        |
| 2.6 An Application of the Graphical Hopf Theorem .....                                | 34        |
| <br>  |           |
| <b>3. Continuation of Bifurcation Curves on the Parameter Plane .....</b>             | <b>43</b> |
| 3.1 Introduction .....  | 44        |
| 3.2 Static and Dynamic Bifurcations .....   | 45        |
| 3.2.1 Formulation of elementary bifurcation conditions .....                          | 45        |
| 3.2.2 Applications of the frequency domain formulas .....                             | 51        |
| 3.2.2.1 The saddle-node bifurcation.....  | 51        |
| 3.2.2.2 The transcritical bifurcation.....  | 52        |
| 3.2.2.3 The hysteresis bifurcation .....  | 54        |
| 3.2.2.4 The pitchfork bifurcation.....  | 56        |
| 3.2.2.5 Static bifurcation in chemical reactor models .....                           | 58        |
| 3.3 Bifurcation Analysis in the Frequency Domain .....                                | 62        |
| 3.3.1 Formulation of multiple crossings and determination of<br>degeneracies .....    | 62        |
| 3.3.2 Applications of the frequency domain formulas to multiple<br>bifurcations ..... | 67        |
| 3.4 Degenerate Hopf Bifurcations of Co-Dimension 1 .....                              | 76        |

|  |            |
|--|------------|
| <b>3.5 Applications and Examples .....</b>   | <b>87</b>  |
| <b>3.5.1 Continuation of bifurcation curves in the reactor with consecutive reactions .....</b>                  | <b>87</b>  |
| <b>3.5.2 Continuation of bifurcation curves in the reactor with extraneous thermal capacitance .....</b>         | <b>96</b>  |
| <b>4. Degenerate Bifurcations in the Space of System Parameters .....</b>  | <b>99</b>  |
| <b>4.1 Introduction .....</b>  | <b>100</b> |
| <b>4.2 Multiplicity of Equilibrium Solutions .....</b>   | <b>102</b> |
| <b>4.3 Multiple Hopf Bifurcation Points .....</b>  | <b>105</b> |
| <b>4.4 Degenerate Hopf Bifurcations and the Singularity Theory .....</b>   | <b>129</b> |
| <b>4.5 Degenerate Hopf Bifurcations and Feedback Systems .....</b>   | <b>140</b> |
| <b>4.6 Degenerate Hopf Bifurcations and the Graphical Hopf Theorem .....</b>                                     | <b>150</b> |
| <b>4.6.1 Degenerate Hopf bifurcations of the <math>H_{0m}</math> type .....</b>                                  | <b>151</b> |
| <b>4.6.2 Degenerate Hopf bifurcations of the <math>H_{n0}</math> type .....</b>                                  | <b>156</b> |
| <b>4.7 Some Applications .....</b>   | <b>163</b> |
| <b>5. High-Order Hopf Bifurcation Formulas .....</b>   | <b>171</b> |
| <b>5.1 Introduction .....</b>  | <b>172</b> |
| <b>5.2 Approximation of Periodic Solutions by Higher-Order Formulas .....</b>                                    | <b>173</b> |
| <b>5.2.1 The algorithm .....</b>   | <b>177</b> |
| <b>5.2.2 Some applications .....</b>   | <b>178</b> |
| <b>5.3 Continuation of Periodic Solutions: Degenerate Cases .....</b>  | <b>191</b> |
| <b>5.4 Local Bifurcation Diagrams and the Graphical Hopf Theorem .....</b>                                       | <b>207</b> |
| <b>5.5 Algorithms for Recovering Periodic Solutions .....</b>  | <b>209</b> |
| <b>5.5.1 The original formulation (OF) .....</b>   | <b>209</b> |
| <b>5.5.2 The modified scheme (MS) .....</b>  | <b>211</b> |
| <b>5.5.3 An iterative algorithm (IA) .....</b>   | <b>211</b> |
| <b>5.6 Multiple Limit Cycles and Numerical Problems .....</b>  | <b>212</b> |
| <b>6. Hopf Bifurcation in Nonlinear Systems with Time Delays .....</b>   | <b>219</b> |
| <b>6.1 Introduction .....</b>  | <b>220</b> |
| <b>6.2 Conditions for Degenerate Bifurcations in Time-Delayed Systems .....</b>                                  | <b>222</b> |
| <b>6.3 Applications in Control Systems .....</b>   | <b>228</b> |
| <b>6.3.1 Variable structure control and Smith's predictor .....</b>  | <b>228</b> |
| <b>6.3.2 Cascade of time-delayed feedback integrators .....</b>  | <b>231</b> |
| <b>6.4 Time-Delayed Feedback Systems: The General Case .....</b>   | <b>240</b> |
| <b>6.5 Application Examples .....</b>  | <b>244</b> |
| <b>6.5.1 Hopf bifurcation in a phase-locked loop circuit with time-delay .....</b>                               | <b>244</b> |
| <b>6.5.2 Hopf bifurcation and degeneracies in a nonlinear feedback control system with two time-delays .....</b> | <b>247</b> |

|  |            |
|--|------------|
| <b>7. Birth of Multiple Limit Cycles .....</b>           | <b>255</b> |
| 7.1 Introduction .....                                   | 256        |
| 7.2 Harmonic Balance and Curvature Coefficients .....    | 258        |
| 7.3 Some Application Examples .....                      | 265        |
| 7.4 Controlling the Multiplicities of Limit Cycles ..... | 273        |
| <b>Appendix .....</b>                                    | <b>275</b> |
| A. Higher-Order Hopf Bifurcation Formulas: Part I .....  | 275        |
| B. Higher-Order Hopf Bifurcation Formulas: Part II ..... | 294        |
| C. Higher-Order Hopf Bifurcation Formulas: Part III..... | 296        |
| <b>References .....</b>                                  | <b>299</b> |
| <b>Author Index .....</b>                                | <b>311</b> |
| <b>Subject Index .....</b>                               | <b>319</b> |