

Contents

Introduction and Overview	1
1 The Spectrum of Linear Operators and Hilbert Spaces	9
1.1 The Spectrum	9
1.2 Properties of the Resolvent	12
1.3 Hilbert Space	13
2 The Geometry of a Hilbert Space and Its Subspaces	17
2.1 Subspaces	17
2.2 Linear Functionals and the Riesz Theorem	20
2.3 Orthonormal Bases	22
3 Exponential Decay of Eigenfunctions	27
3.1 Introduction	27
3.2 Agmon Metric	30
3.3 The Main Theorem	31
3.4 Proof of Theorem 3.4	33
3.5 Pointwise Exponential Bounds	34
3.6 Notes	37

4 Operators on Hilbert Spaces	39
4.1 Remarks on the Operator Norm and Graphs	39
4.2 The Adjoint of an Operator	40
4.3 Unitary Operators	45
5 Self-Adjoint Operators	49
5.1 Definitions	49
5.2 General Properties of Self-Adjoint Operators	51
5.3 Determining the Spectrum of Self-Adjoint Operators	53
5.4 Projections	56
6 Riesz Projections and Isolated Points of the Spectrum	59
6.1 Riesz Projections	59
6.2 Isolated Points of the Spectrum	64
6.3 More Properties of Riesz Projections	65
6.4 Embedded Eigenvalues of Self-Adjoint Operators	67
7 The Essential Spectrum: Weyl's Criterion	69
7.1 The Weyl Criterion	69
7.2 Proof of Weyl's Criterion: First Part	71
7.3 Proof of Weyl's Criterion: Second Part	72
8 Self-Adjointness: Part 1. The Kato Inequality	77
8.1 Symmetric Operators	77
8.2 Fundamental Criteria for Self-Adjointness	79
8.3 The Kato Inequality for Smooth Functions	81
8.4 Technical Approximation Tools	82
8.5 The Kato Inequality	84
8.6 Application to Positive Potentials	85
9 Compact Operators	89
9.1 Compact and Finite-Rank Operators	89
9.2 The Structure of the Set of Compact Operators	91
9.3 Spectral Theory of Compact Operators	93
9.4 Applications of the General Theory	96
10 Locally Compact Operators and Their Application to Schrödinger Operators	99
10.1 Locally Compact Operators	99
10.2 Spectral Properties of Locally Compact Operators	101
10.3 Essential Spectrum and Weyl's Criterion for Certain Closed Operators	104
11 Semiclassical Analysis of Schrödinger Operators I: The Harmonic Approximation	109
11.1 Introduction	109

11.2 Preliminary: The Harmonic Oscillator	110
11.3 Semiclassical Limit of Eigenvalues	112
11.4 Notes	117
12 Semiclassical Analysis of Schrödinger Operators	
II: The Splitting of Eigenvalues	119
12.1 More Spectral Analysis: Variational Inequalities	119
12.2 Double-Well Potentials and Tunneling	121
12.3 Proof of Theorem 12.3	124
12.4 Appendix: Exponential Decay of Eigenfunctions for Double-Well Hamiltonians	127
12.5 Notes	128
13 Self-Adjointness: Part 2. The Kato–Rellich Theorem	131
13.1 Relatively Bounded Operators	131
13.2 Schrödinger Operators with Relatively Bounded Potentials	136
14 Relatively Compact Operators and the Weyl Theorem	139
14.1 Relatively Compact Operators	140
14.2 Weyl’s Theorem: Stability of the Essential Spectrum	141
14.3 Applications to the Spectral Theory of Schrödinger Operators	143
14.4 Persson’s Theorem: The Bottom of the Essential Spectrum	145
15 Perturbation Theory: Relatively Bounded Perturbations	149
15.1 Introduction and Motivation	149
15.2 Analytic Perturbation Theory for the Discrete Spectrum	150
15.3 Criteria for Eigenvalue Stability: A Simple Case	151
15.4 Type-A Families of Operators and Eigenvalue Stability: General Results	153
15.5 Remarks on Perturbation Expansions	156
15.6 Appendix: A Technical Lemma	157
16 Theory of Quantum Resonances	
I: The Aguilar–Balslev–Combes–Simon Theorem	161
16.1 Introduction to Quantum Resonance Theory	161
16.2 Aguilar–Balslev–Combes–Simon Theory of Resonances	164
16.3 Proof of the Aguilar–Balslev–Combes Theorem	168
16.4 Examples of the Generalized Semiclassical Regime	171
16.5 Notes	173
17 Spectral Deformation Theory	177
17.1 Introduction to Spectral Deformation	177
17.2 Vector Fields and Diffeomorphisms	178
17.3 Induced Unitary Operators	180
17.4 Complex Extensions and Analytic Vectors	181
17.5 Notes	186

18 Spectral Deformation of Schrödinger Operators	187
18.1 The Deformed Family of Schrödinger Operators	187
18.2 The Spectrum of the Deformed Laplacian	190
18.3 Admissible Potentials	192
18.4 The Spectrum of Deformed Schrödinger Operators	193
18.5 Notes	195
19 The General Theory of Spectral Stability	197
19.1 Examples of Nonanalytic Perturbations	198
19.2 Strong Resolvent Convergence	200
19.3 The General Notion of Stability	202
19.4 A Criterion for Stability	203
19.5 Proof of the Stability Criteria	207
19.6 Geometric Techniques and Applications to Stability	210
19.7 Example: A Simple Shape Resonance Model	212
20 Theory of Quantum Resonances	
II: The Shape Resonance Model	215
20.1 Introduction: The Gamow Model of Alpha Decay	215
20.2 The Shape Resonance Model	216
20.3 The Semiclassical Regime and Scaling	218
20.4 Analyticity Conditions on the Potential	221
20.5 Spectral Stability for Shape Resonances: The Main Results	223
20.6 The Proof of Spectral Stability for Shape Resonances	227
20.7 Resolvent Estimates for $H_1(\lambda, \theta)$ and $H(\lambda, \theta)$	229
20.8 Notes	232
21 Quantum Nontrapping Estimates	235
21.1 Introduction to Quantum Nontrapping	235
21.2 The Classical Nontrapping Condition	238
21.3 The Nontrapping Resolvent Estimate	241
21.4 Some Examples of Nontrapping Potentials	247
21.5 Notes	249
22 Theory of Quantum Resonances	
III: Resonance Width	251
22.1 Introduction and Geometric Preliminaries	251
22.2 Exponential Decay of Eigenfunctions of $H_0(\lambda)$	253
22.3 The Proof of Estimates on Resonance Positions	257
23 Other Topics in the Theory of Quantum Resonances	263
23.1 Stark and Stark Ladder Resonances	263
23.2 Resonances and the Zeeman Effect	272
23.3 Resonances of the Helmholtz Resonator	275
23.4 Comments on More General Potentials, Exponential Decay, and Lower Bounds	280

Appendix 1. Introduction to Banach Spaces	285
A1.1 Linear Vector Spaces and Norms	285
A1.2 Elementary Topology in Normed Vector Spaces	286
A1.3 Banach Spaces	288
A1.4 Compactness	291
Appendix 2. The Banach Spaces $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$	293
A2.1 The Definition of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$	293
A2.2 Important Properties of L^p -Spaces	296
1. Density results	296
2. The Hölder Inequality	297
3. The Minkowski Inequality	298
4. Lebesgue Dominated Convergence	299
Appendix 3. Linear Operators on Banach Spaces	301
A3.1 Linear Operators	301
A3.2 Continuity and Boundedness of Linear Operators	303
A3.3 The Graph of an Operator and Closure	307
A3.4 Inverses of Linear Operators	309
A3.5 Different Topologies on $\mathcal{L}(X)$	312
Appendix 4. The Fourier Transform, Sobolev Spaces, and Convolutions	313
A4.1 Fourier Transform	313
A4.2 Sobolev Spaces	316
A4.3 Convolutions	317
References	319
Index	333